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THE HISTORY OF MATHEMATICS.

A Short Account of the History of Mathematics. By W. W. Rouse Ball. (London and New York: Macmillan and Co., 1888.)

THE quaint words addressed "to the great variety of readers" by the editors of the folio Shakespeare of 1623 are equally applicable to the useful compendium of mathematical history which is the subject of our review. "It is now public; and you will stand for your privileges, we know—to read and censure. Do so, but buy it first: that doth best commend a book, the stationer says. Then how odd soever your brains be or your wisdoms, make your licence the same, spare not." But, as goods are usually "bought by judgment of the eye, not uttered by base sale of chapmen's tongues," we produce our samples in the open market by making a few extracts from Mr. Ball's book.

In the opening chapter, on Egyptian and Phœnician mathematics, we become acquainted with an old Egyptian, "a priest named Ahmes," who, "somewhere between the years 1700 B.C. and 1100 B.C.," wrote, on imperishable papyrus, a book entitled "Directions for Knowing all Dark Things," which "is believed to be itself a copy with emendations, of an older treatise of about the time 3400 B.C." Remembering that this work was written certainly five hundred, and probably more than a thousand, years before the time of Thales, the first of the Greek mathematicians, and founder of the Ionian school, it must be regarded as a most remarkable production; for Profs. Cantor and Eisenlohr have shown that Ahmes had some notion of trigonometry. In his problems on pyramids, "Ahmes desires to find the ratio of certain lines, which is equivalent to determining the trigonometrical ratios of certain angles. The data and the results given agree closely with the measurements of some of the existing pyramids." But perhaps the most interesting feature of this ancient treatise is the algebraic notation employed in it, which our author describes in these words:—"The unknown quantity is always represented by the symbol which means a heap; addition is represented by a pair of legs walking forwards; subtraction by a pair of legs walking backwards, or by a flight of arrows; and equality by the sign /∠." Our own + and – first appeared in Widman's "Mercantile Arithmetic" (published at Leipzig in 1489): with him (see p. 186, Ch. XII.) they "are only abbreviations, and not symbols of operation; he attached little or no importance to them, and would no doubt have been amazed if he had been told that their introduction was preparing the way for a complete revolution of the processes used in algebra." The philosophic conception of the nature of algebra (symbolized by the legs walking forwards and backwards; a notion closely related to, if not identical with, Sir W. R. Hamilton's definition of algebra as the science of pure time) perished with its author: the mere abbreviations (+ and –) lived and flourished—but then Widman was able to print *his* book.

The first date that can be assigned with absolute precision is that of Thales. "It is well known that he predicted a solar eclipse which took place at or about the time he foretold: the actual date was May 28, 585 B.C." It marks the real commencement of the history of mathematics; for the science, now revived in Greece, was at this time neglected and completely forgotten by the Egyptians. When we read that Thales, to the utter amazement of the King and all who were present, showed them how to find the height of a pyramid, by a simple application of the theorem that *the sides of equiangular triangles are proportionals*, we may well wonder why Ahmes did not burst his mummy-case and appear in their midst with his book opened at the problems on pyramids.

From the time of Thales to that of Euclid, the knowledge of mathematical facts acquired in one generation was transmitted to the next, almost exclusively by means of oral tradition. That such was the case is mainly due to the Pythagorean secret Society. "Pythagoras himself did not allow the use of text-books, and the assumption of his school was, not only that all their knowledge was held in common, and secret from the outside world, but that the glory of any fresh discovery must be referred back to their founder: thus Hippasus (*circ.* 470 B.C.) is said to have been drowned for violating his oath by publicly boasting that he had added the dodecahedron to the number of regular solids enumerated by Pythagoras. Gradually, as the Society became more scattered, it was found convenient to alter this rule, and treatises containing the substance of their teaching and doctrines were written. The first book of the kind was composed by Philolaus (*circ.* 410 B.C.), and we are told that Plato contrived to buy a copy of it."

Now Anaximander, the immediate successor of Thales as head of the Ionian school, had the honour of teaching Pythagoras; while Eudoxus, Philolaus, and Plato, all of them received their mathematical training from Archytas of Tarentum, who was one of the most celebrated of the Pythagoreans; and "Menæchmus, who was a pupil of Plato and Eudoxus," was alive as late as 325 B.C., which brings us down to about the time of Euclid. Thus the chain of tradition connecting Thales with Euclid is complete. Its successive links can be traced in the second and third chapters of the work before us.

Among the contemporaries of Plato, Eudoxus of Cnidus deserves special notice. His biography is to be found in Diogenes Laertius, who speaks of him as an astronomer, geometer, physician, and statesman; mentions his great works on astronomy and geometry, and his minor treatises on other subjects; and refers to the fact that he *discovered curved lines.* Modern research has found out what the curves of Eudoxus were, though all his writings are lost: in our author's words, "he discussed some plane sections of the anchor ring," among them the curve which ought in future to be named after him, but is "generally called Bernouilli's lemniscate." Thus, Eudoxus (who died in 355 B.C.) anticipated James Bernouilli (d. 1705 A.D.) by more than 2000 years!

The foundation of Alexandria by Ptolemy marks an epoch in the history of mathematics. Alexander himself did little more than choose the site, and it was entirely due to Ptolemy that the city did not share the fate of at least two others of the same name whose foundation by Alexander is duly recorded by his biographer, Quintus Curtius. What Alexandria actually became, is thus briefly and graphically described :—

> "The earliest attempt to found a University, as we understand the word, was made at Alexandria. Richly endowed, supplied with lecture-rooms, libraries, museums, laboratories, gardens, and all the plant and machinery that ingenuity could suggest, it became at once the intellectual metropolis of the Greek race, and remained so for a thousand years. It was particularly fortunate in producing, within the first century of its existence, three of the greatest mathematicians of antiquity—Euclid, Archimedes, and Apollonius. They laid down the lines on which mathematics were subsequently studied, and, largely owing to their influence, the history of mathematics centres more or less round that of Alexandria, until the destruction of the city by the Arabs in 641 A.D."

It would occupy too much space to discuss, or even to enumerate, the writings of the Alexandrian mathematicians. The most precious relics they have left behind them are : the greater part of the numerous works of Euclid, many of the writings of Archimedes, the "Conics" of Apollonius, the "Almagest" of Ptolemy, the "Mathematical Collections" of Pappus, and the "Arithmetic," or, rather, the "Algebra," of Diophantus. These and other valuable pieces of work, which, like them, have reached us in a more or less mutilated condition, are reviewed in the fourth and fifth chapters of Mr. Ball's "History," in which the best editions of these classical authors are mentioned, and other sources of information concerning them are referred to. We owe the preservation of most of them to the Greek refugees at Constantinople, as will be seen from the following quotation :—

> "After the capture of Alexandria by the Mohammedans, the majority of the philosophers, who had previously been teaching there, migrated to Constantinople, which then became the centre of Greek learning in the East, and remained so for 900 years. But, though its history covers such an immense interval of time, it is utterly barren of any scientific interest; and its chief merit is that it preserved for us the works of the different Greek schools. The revelation of these works to the West in the fifteenth century was one of the most important sources of the stream of modern European thought, and the history of the school may be summed up by saying that it played the part of a conduit-pipe in conveying to us the results of an earlier and brighter age."

Before the fall of Constantinople in 1453, which is alluded to in the above extract, such mathematics as were known in Western Europe were derived from Arabian sources.

The history of Arab mathematics and their introduction into Europe forms the subject-matter of the ninth and tenth chapters of Mr. Ball's book. The first of these excellent chapters tells us, in the beginning, how the Arabs, by their intercourse with Constantinople and India, in the reign of Al Mamun, the successor of the renowned Caliph Haroun Al Raschid, acquired a knowledge of the principal Greek and Hindu authors ; it then gives an account of the works of the three chief Hindu mathematicians, Arya-Bhatta, Brahmagupta, and Bhaskara ; and finishes with an analysis of the great treatise of Alkarismi, the first Arab mathematician, and an enumeration of the works of the most prominent among his successors from Tabit-ibn-Korra down to Alhazen and Abd-el-gehl. The account of Bhaskara is very much fuller than that given by M. Maximilien Marie in his "Histoire des Sciences Mathématiques et Physiques" (twelve vols. 8vo, 1883–88), and in other parts of the chapter some very interesting facts are mentioned, which we do not find noticed by M. Marie. Among these we may instance the solution of the cubic by Tabit-ibn-Korra, about 650 years before the time of Tartaglia, and, what is even more remarkable, the enunciation by Alkhodjandi of the proposition that the sum of two cubes can never be a cube.

The next chapter begins with the introduction of mathematics into Europe by the Moorish conquerors of Spain in the eighth century ; shows how the Christians gained from them some knowledge of Arab science in the twelfth century, and, before the end of the thirteenth, were in possession of "copies of Euclid, Archimedes, Apollonius, Ptolemy, and some of the Arab works on algebra" ; and brings the history of European mathematics down to the middle of the fifteenth century. During this long interval there lived only two great mathematicians in all Christendom, both of whom belonged to the thirteenth century. One was the famous Roger Bacon ; the other, Leonardo Fibonacci, of Pisa, was the earliest European writer on algebra that we are acquainted with. Their biographies, though concisely written, necessarily occupy a large portion of the chapter.

The three following chapters contain the history of mathematics from the invention of printing to the year 1637, when the "Géométrie" of Descartes made its appearance. In this brief space of time, barely three-quarters of a century, owing to the labours of Pacioli, Recorde, Stifel, Tartaglia, Cardan, Ferrari, Bombelli, Vieta, Harriot, Oughtred, Stevinus, and others, vast improvements in algebra had been effected ; trigonometrical and logarithmic tables had been brought to a high state of perfection by Regiomontanus, Rheticus, Napier, and Briggs ; Desargues had invented the modern projective geometry ; while, in astronomy, Copernicus, Kepler, and Galileo had replaced the old Ptolemaic system by a still older one (propounded by the Pythagoreans), which was now, for the first time, established on a firm basis.

Our author, as he tells us in the preface, has "usually omitted all reference to practical astronomers, unless there is some mathematical interest in the theories they proposed," and, accordingly, the name of Tycho Brahe does not figure in the above list. It would be better, in our opinion, to treat Copernicus in the same manner, rather than to do him the injustice of speaking of "his conjecture that the earth and planets revolved round the sun." Granting that "he advocated it only on the ground that it gave a simple explanation of natural phenomena," we would ask what other, or what better, proof could he have of it? It should be borne in mind that Copernicus spent the best years of his life in testing his "conjecture" by observations, and that nothing short of a firm conviction of its truth could possibly have induced him to publish it in the face of the fierce opposition which he well knew it would provoke.

With this exception, the short sketches of the lives and writings of all the mathematicians we have named are well drawn, and convey a clear idea of the importance of their work, and of the amount contributed by each of them to the advancement of the science.

The remaining portion—about half—of the book is divided into six chapters (numbered XIV. to XIX. inclusive), in which the history of modern mathematics is briefly considered. These are so full of great discoveries and illustrious names that they must be read to be appreciated. We can only, in the limited space at our disposal, quote their titles and add some remarks.

Chapter XIV. "Features of Modern Mathematics." In this chapter, which is a sort of summary of the other five, we read that "five distinct stages in the history of this period can be discerned." Turning to the table of contents, we find the five stages thus described: (1) "invention of analytical geometry and the method of indivisibles," (2) "invention of the calculus," (3) "development of mechanics," (4) "application of mathematics to physics," (5) "recent development of pure mathematics." The mere remark that each of these might be made the title of a bulky volume, will show at once the enormous extent and importance of modern mathematics.

Chapter XV. "History of Mathematics from Descartes to Huygens." The principal names in this chapter are Descartes, Cavalieri, Pascal, Wallis, Fermat, Barrow, and Huygens. In many of their writings may be found the germs of those ideas which have since been developed in the infinitesimal calculus. Especially would we mention Cavalieri's *method of indivisibles*, of which our integral calculus is the modified descendant, and Barrow's method of drawing tangents to curves, substantially the same as that given at the beginning of any modern differential calculus. Full explanations of both methods may be found in the present chapter.

The history of modern mathematics dates from the publication of the "Géométrie" of Descartes, and we wish to call attention to a bibliographical point connected with it. M. Marie ("Histoire," &c., t. iv. p. 20) speaks of "quatre traités separés: 'Le Discours de la Méthode,' 'La Dioptrique,' 'Les Météores,' et 'La Géométrie,'" all of them published in 1637; Mr. Ball (p. 241) says that "Descartes's researches in geometry are given in the third section of the 'Discours.'" We cannot positively say which is correct, but our impression is that we have seen a copy of the separately-published "Géométrie." The point is of small importance, but it should be cleared up in subsequent editions.

Chapter XVI. "The Life and Works of Newton." There are two sections—one devoted to the life, the other to an analysis of the works, of our English Archimedes; his three capital discoveries—fluxions, the decomposition of light, and universal gravitation—will occur to most of our readers. Most of the well-known facts relating to Newton's private and public life are mentioned in this chapter, together with some others that have only recently come to light.

Chapter XVII. "Leibnitz and the Mathematicians of the First Half of the Eighteenth Century." The following sentence occurs in the opening paragraph:—

> "Modern analysis is, however, derived directly from the works of Leibnitz and the elder Bernouillis; and it is immaterial to us whether the fundamental ideas of it were obtained by them from Newton, or discovered independently."

It forms a fitting sequel to the tale told in the preceding chapter of the celebrated controversy between Newton and Leibnitz.

The present chapter is in three sections: (1) "Leibnitz and the Bernouillis," (2) "The Development of Analysis on the Continent," (3) "The English Mathematicians of the Eighteenth Century." The two greatest French names in the chapter are those of Clairaut and D'Alembert; the two greatest English ones, those of Taylor and Maclaurin. Matthew Stewart succeeded Maclaurin as Professor at Edinburgh, and was "almost the only other British writer of any marked eminence in pure mathematics during the eighteenth century." After recounting his chief works, our author proceeds to say:—

> "These prove him to have been a mathematician of great natural power, but, unfortunately, he followed the fashion set by Newton and Maclaurin, and confined himself to geometrical methods."

This sentence gives the history, in epitome, of the decline and fall of British mathematics in the last century.

Chapter XVIII. "Lagrange, Laplace, and their Contemporaries." There are four sections: (1) "The Development of Analysis and Mechanics," (2) "The Creation of Modern Geometry," (3) "The Development of Mathematical Physics," (4) "The Introduction of Analysis into England." The greatest foreign name in this chapter (we single it out from a number of others) is that of Euler; the greatest English one is possibly that of Thomas Simpson, who seems to be rather harshly treated by being allotted only three lines in a footnote, when others of less ability are noticed in the text.

In Section 4 we read: "The introduction of the notation of the differential calculus into England was due to three undergraduates at Cambridge—Babbage, Peacock, and Herschel—to whom a word or two may be devoted."

Doubtless the success of the movement was largely due to their efforts, but the initiative was taken by Woodhouse in 1803 (see J. W. L. Glaisher on the "Tripos," Proc. Lond. Math. Soc., vol. xviii. p. 18). The name of Woodhouse is surely as deserving of mention as the other three.

Chapter XIX. "Recent Times." The author begins with a long list of names well known in the mathematical world. This list, however, "is not and does not pretend to be exhaustive." He then classifies the writers he has enumerated "according to the subjects in connection with which they are best known, arranging the latter in the following order: elliptic and Abelian functions, theory of numbers, higher algebra, modern geometry, analytical geometry, analysis, astronomy, and physics."

The section on the theory of numbers is, in our opinion, the best. It contains biographies of Gauss and the late Prof. Smith (about four pages being allotted to each), and mentions the researches of Cauchy, Liouville, Eisenstein, Kummer, Kronecker, Hermite, Dedekind, and Tchebycheff.

We may now say with old Martial—

> "Ohe jam satis est, ohe libelle:
> Jam pervenimus usque ad umbilices."

But we have yet to record the impression left by the perusal of the entire work. The most desirable thing in a book of reference is that the reader should be enabled to find his way readily to any part of it. In the one before us this want is met by an admirable index, and an equally complete table of contents, and by the liberal use of clarendon type in the body of the book. The printing is clear and generally correct, but we notice the following errata:—

P. x. line 8 from top, for "1885–1888" read "1883–1888."

P. 110, in the heading of Chapter VI., for "641–1543" read "641–1453."

P. 168, line 4 from bottom, for "Act iv. sc. 3" read "Act iv. sc. 2."

P. 358, line 8 from bottom, for "1728" read "1738."

All the salient points of mathematical history are given, and many of the results of recent antiquarian research; but it must not be imagined that the book is at all dry. On the contrary, the biographical sketches frequently contain amusing anecdotes, many of the theorems mentioned are very clearly explained, so as to bring them within the grasp of those who are only acquainted with elementary mathematics, and there is a very interesting account (in a footnote) of the early history of the Universities of Paris, Oxford, and Cambridge. For those who wish to study mathematical history in detail there is a long list of authorities at the beginning, and many references to other works are made in different parts of the book. We would suggest that in future editions reference should be made to "Les Fondateurs de l'Astronomie Moderne Copernic—Tycho Brahé, Képler, Galilée, Newton," by Joseph Bertrand (8vo, Paris, n.d.), and to the article "Viga Ganita" in the "Penny Cyclopædia" (which contains the opinions of Colebrooke, the translator of the "Lilavati," &c., on many points connected with Hindu mathematics).

Finally, we would suggest that the following motto should be printed on the title-page of the second edition:—

> "Habetis originis ac progressionis mathematicæ historiam brevem. Ex qua matheseos antiquitas, præstantia, ac dignitas apparet."

The quotation is taken from the concluding paragraph of the "Historica Narratio" prefixed to Andrew Tacquet's "Euclid" (2nd ed., by Whiston, 1710). It describes perfectly the contents of the present treatise.

Mr. Ball promises us a supplementary volume containing a list of mathematicians and their works, which is to be as complete as possible. It will be a most important contribution to mathematical bibliography, and we sincerely hope that the reception that this volume meets with will encourage him to write the supplement.

THE BUILDING OF THE BRITISH ISLES.

The Building of the British Isles: a Study in Geographical Evolution. By A. J. Jukes-Browne, B.A., F.G.S. (London: George Bell and Sons, 1888.)

IT is now thirty-three years since Godwin-Austen, in a paper which glows with the instinctive perception that is one of the marks of genius, suggested to geologists an application of their science which lifts it out of the region of technicalities, gives it a human interest, and attracts all those who care to follow the long chain of events of which the present state of things is the outcome. It was an attempt to go back to Mesozoic and Palæozoic days, and mark out the main outlines of the physical geography of Great Britain and the adjoining parts of Europe during those epochs. To enable its conclusions to be more easily grasped, the paper was accompanied by a map, almost bewildering in its complexity and somewhat hazy in its outlines, but full of the masterly generalization that marshals into one compact body a crowd of isolated facts, and of the intuition that foresees the complete meaning of imperfectly ascertained data.

Many a geologist has since been tempted to try his hand at similar tasks, but few have mustered courage, when it came to the point, to embody their conclusions in a map. And no wonder: everyone who has speculated in this direction knows how easy it is to clothe his conceptions in words, and soon finds out how hard verbal descriptions of physical geography are to follow. So he becomes keenly alive to the fact that, if he wishes to be listened to, he must make the road easy by presenting his restorations to the eye in the pictoral form of a map. But if he be haunted by any sense of accuracy, and any horror of vagueness and hasty reasoning, he finds himself beset on all sides, when he begins to plot out his map, with uncertainties and hesitations that give him pause. It may be easy to say that land lay on this side and sea on that, but when a coast-line is actually to be laid down, though it may be possible to fix the limits between which it must lie, these limits are often so wide apart that the feeling of uncertainty as to the actual position of the boundary becomes unbearable, and the prospect of making a map that shall be even approximately accurate grows hopeless. Worse still is it—and this not unfrequently happens—when there are not even bounding limits, and the coast-lines can be no better than such guess-work as rashness delights in and the logical temperament abhors.

But even those who realize most clearly the difficulties of the task of making maps which show the distribution of land and sea during past geological epochs, welcome with keen delight attempts, such as those in the book before us, which are made in the right spirit; and it would ill become me to carp at the author's restorations, even were they less satisfactory than is the case, for I believe that, in noticing a former work of his, I ventured to take him to task for not having appended maps to his verbal descriptions of the old physical geography of our islands.

Mr. Jukes-Browne has explained, in the introduction, the principles which have been his guide; and the words with which he concludes his opening remarks show how fully he is aware of the difficulties that attend the task he has undertaken, and how much uncertainty hangs over many of his results. Even where we cannot agree with him, we feel sure that he has never been hasty and has spared no pains to arrive at the most probable conclusions.

With commendable caution no attempts are made to depict on a map the physical geography of Archæan and Cambrian times; but preference is given to Prof. Hull's conjecture that the great mass of Cambrian land "lay to the north-west of Europe, and occupied a large part of what is now the North Atlantic Ocean." The words "large part" are vague, but a partiality for filling up the